Technical Note

Self-Reinforcing Dynamics

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Why is Madonna so popular whereas other singers with similar voice and looks never made it past the local bar scene? Why did VHS become the dominant consumer video-cassette recorder design, whereas Sony's Betamax fell into oblivion? To be sure, many different factors determine the fate of stars and technologies. In this note, we look into one particular factor: in some markets, self-reinforcing dynamics — to give more to those who already have — lead to winner-take-all or winner-take-most outcomes of the sort we observe with stars and technologies.

Coins, urns, and the Law of Large Numbers

Suppose that you flip a coin one hundred times. What percentage of those tosses will be heads? If you are flipping a fair coin, chances are it will be fairly close to 50%: sometimes a little higher, sometimes a little lower, but always around 50%. In fact, one of the most important results in statistics is the *Law of Large Numbers*. Basically it states that, if you keep tossing a coin indefinitely, then the fraction of times it comes out heads will almost surely converge to one half. (The term "almost surely" has a very precise meaning in statistics, but we need not get into it at this time.)

Let us now move from coins to urns. Specifically, consider an urn with one red ball and one blue ball. Take one ball from the urn at random and note its color. Place the ball back in the urn. Repeat the process and count the number of blue balls you get after one hundred tries. What is the percentage of blue balls likely to be?

From a statistical point of view, the above urn process is the same as tossing a coin: the probability of extracting a blue ball is the same as the probability of getting a head toss: one half. It follows that the percentage of blue balls follows the same dynamic stochastic process as the percentage of heads. By dynamic stochastic process we mean a variable (e.g., percentage of blue balls) that evolves over time according to some random series of events.

The left panel in Exhibit 1 illustrates the dynamics of blue-ball market shares. The three lines correspond to three possible stochastic paths. Notice that, while there is considerable variance in the early stages of the process (both within a given path and across paths), as time goes by the percentage of blue balls tends toward 50%. This is the law of large numbers in practice.[1]

Written by Professor Luís Cabral for the purpose of class discussion rather than to illustrate either effective or ineffective handling of an administrative situation.

Polya urn schemes

The above urn process is interesting, but very quickly it becomes boring. Consider now a more exciting possibility. Suppose that, after you extract a ball from the urn and take note of its color, you place it back in the urn *together with another ball of the same color*. This implies that, starting in period $t = 1$ with an urn containing two balls (one red and one blue), by period t the urn already contains $t + 1$ balls in it.

As before, we ask the question: how does the percentage of blue balls evolve over time? At a given moment in time, when there are b blue balls and r red balls, the probability of extracting a blue ball is given by $b/(b + r)$. Notice this probability changes over time, whereas before it was fixed at 1/2. The law of large numbers, which requires a constant probability, no longer applies.

The answer is the following: the percentage of blue balls will almost surely converge to some value (just as in the previous urn process). However, the value it converges to need not be equal to 1/2. In fact, it could be *any* value between 0 and 1 (each with equal probability).

The right-hand panel in Exhibit 1 illustrates this idea by plotting three different random paths. Following an initial period of great variability (just like in the "normal" urn case), we see that the percentage of blue balls converges to a value, that is, the path flattens out. However, the particular value it converges to varies from run to run.[2]

□ **Generalized Polya schemes.** The above Polya scheme has a very specific property: the probability that a blue ball is added at a given stage is *exactly* equal to the proportion of blue balls inside the urn. This makes sense if we think exclusively about urns with balls that are identical except for their color. However, there may be more complicated processes where the probability that a blue ball is extracted depends on the fraction of blue balls inside the urn.

The general situation is illustrated in Exhibit 2. The solid line exemplifies a general function indicating how the probability of adding a blue ball is related to the current fraction of blue balls inside the urn. What can we say about the limiting fraction of blue balls? Statisticians B. Arthur, Y. Ermoliev and Y. Kanievski have shown that the fraction of blue balls converges almost surely to one of the values where the solid line crosses the diagonal (the shaded line in Exhibit 2) "from above," in the present case the two points market with a bullet point.[3]

Recall that before we considered the case when the probability of adding a blue ball is exactly equal to the fraction of blue balls. In that case, the probability function crosses the diagonal at every point, thus the result that the fraction of blue balls can converge to any value between 0 and 1. We can now see that the simple Polya urn scheme we considered is a bit of a fluke: generically, there is a limited number of points where the function crosses the diagonal.

□ **Ergodicity.** Think of a bowling alley with the particular characteristics that the lanes are (a) very long and (b) convex (instead of being flat, as most bowling alley lanes are — or are supposed to be). For lousy players like myself, this would be great: no matter how I throw the ball, I know that eventually it will find its way to the middle of the lane. We then say the bowling ball follows an ergodic path: its eventual trajectory (which in this case is the middle of the lane) does not depend on what its initial path looked like. Formally, we say that a stochastic process is ergodic if its position at time t does not depend on its position at time $t - \Delta$ for large enough Δ. In other words, in ergodic processes old history is irrelevant.

Consider now an even stranger bowling alley lane: it's as long as the previous one, but instead of being convex this one is concave (like a speed bump looked at sideways). This is one tough alley to bowl at: no matter how accurately you throw the ball down the middle of the lane, it will only be a matter of time before the ball strays from its central trajectory and veers off to the right or to the left — most likely going all the way to the side and staying there forever.

The concave-lane bowling alley is an example of a fundamentally non-ergodic process. We know the ball's trajectory will end up either on the right or on the left end of the lane. If we throw the ball down the middle, the probability that each eventual trajectory unfolds is 50% each. The actual trajectory will most likely be determined by some *small historical event* — a grain of sand or a soft breeze or a small imperfection in the lane's surface — which starts a process of *self-reinforcing* dynamics: once the ball is slightly to the left, gravity will pull is further to the left.

Technology designs and rock stars

What does this all have to do with the real world? Let us go back to one of the examples we started with, video cassette recorders (VCRs). To cut a long story short: Sony's Betamax and JVC's VHS products started off more or less at the same time (mid 1970s) as alternative, incompatible designs for a given new technology (consumer VCRs). Though the models were of approximately comparable quality, and initially sustained comparable market shares, eventually VHS came to dominate the market.

Many articles and books have been written about this interesting case.[4] Many explanations have been proposed to account for VHS's eventual dominance. One common and reasonable explanation relates to the self-reinforcing dynamics we've been talking about. Recall that, from the early 1980s on, most consumers used VCRs to watch movies at home. (In fact, the VCR essentially created the home-video industry as we know it today.) For a while, video-rental stores stocked titles in both formats (VHS and Betamax). However, as Betamax's market share fell below 50%, video-rental stores began to stock fewer Betamax titles than VHS titles. This in turn implied that consumers started to avoid buying Betamax machines, further lowering its market share. And so on.

Exhibit 3 shows the evolution of output levels and market shares in the VCR industry. The vertical dashed lines mark November 1981, the date when the Video Software Dealers Association (VSDA), a trade association for video retailers, was formed. The first video-rental stores were open as early as in December 1977, but the creation of the trade association is a more accurate point to determine the beginning of the video-rental industry.[5] The graphs in Exhibit 3 show that, before the video-rental area, output levels were low and market shares close to 50%. After video-rentals became the primary use for VCRs, total output level increased rapidly, but only one of the standards grew at a significant rate. These dynamics are consistent with a switch from a process like the left-hand panel in Exhibit 1 (an urn with simple replacement) to a process like the right-hand panel in Exhibit 1 (a Polya urn scheme).

What about Madonna? Do Polya urn schemes have anything to say about the emergence of rock stars? Again, many factors are surely at play. But one important determinant in stardom is buzz: people hear about new artists mostly from talking to other people: at work, at school, on Facebook; in a word, in offline and online social networks. We can think of an individual pair interaction in a social network as akin to extracting a ball from an urn: just as different balls come in different colors, so my contacts and friends have different opinions and information about music. And more often than not I will be influenced by them. The process of being influenced, in turn, is akin to replacing a ball in the urn and adding another ball of the same color.

Once we agree that social interaction and influence has — at least to some extent — the features of an urn with replacement, then we can understand how, starting with different artists of relatively similar talents, one of them will achieve considerably greater success than the other — just as the fraction of blue balls may end up at 90% even though originally there was an equal number of blue and red balls inside the urn.

Endotes

1. For aficionados: By the Central Limit Theorem, the distribution of the fraction of blue balls itself converges to a normal distribution, with smaller and smaller variance as the number of coin tosses (or ball extractions) increases. See the technical note, *Black Swans, Fat Tails, and Movie Revenues*, for more details.

2. For aficionados: In the particular case when there are balls of only two colors, any asymptotic share takes place with equal probability. In the more general case when there are colors on n types, the probability distribution of limiting shares is given by $(n-1)\,(1-s)^{n-2}$, where s is the share of a given color.

3. W. Brian Arthur, Yu. M. Ermoliev, and Yu. M. Kaniovski, "Strong Laws for a Class of Path-Dependent Urn Processes," in Proc. International Conf. on Stochastic Optimization, Kiev 1984, Springer, Series Info. and Control, 81, 1986.

4. For a very brief description of the case, see Chapter 17 of Luís Cabral, *Introduction to Industrial Organization*, MIT Press, 2000.

5. Source: The Entertainment Merchant Association, http://entmerch.org/industry_history.html, accessed October 16, 2009.

Exhibit 1

Dynamic stochastic processes under different assumptions.

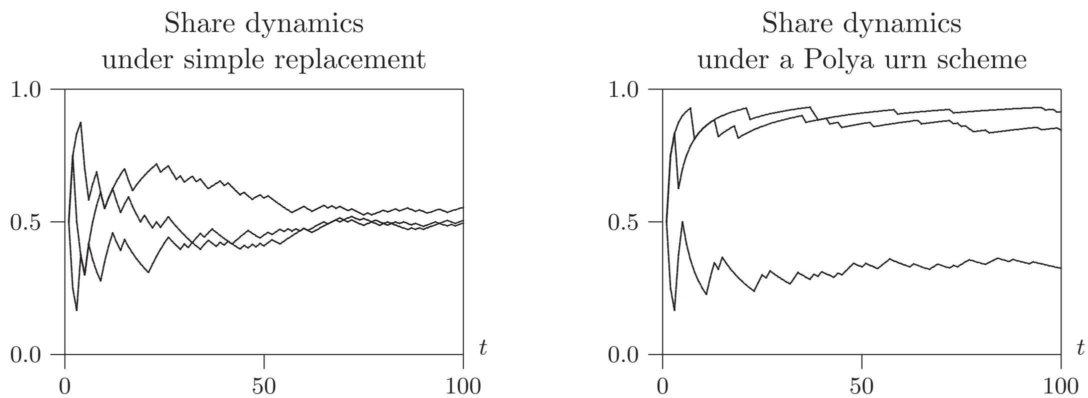

Exhibit 2

Event probability and limiting share.

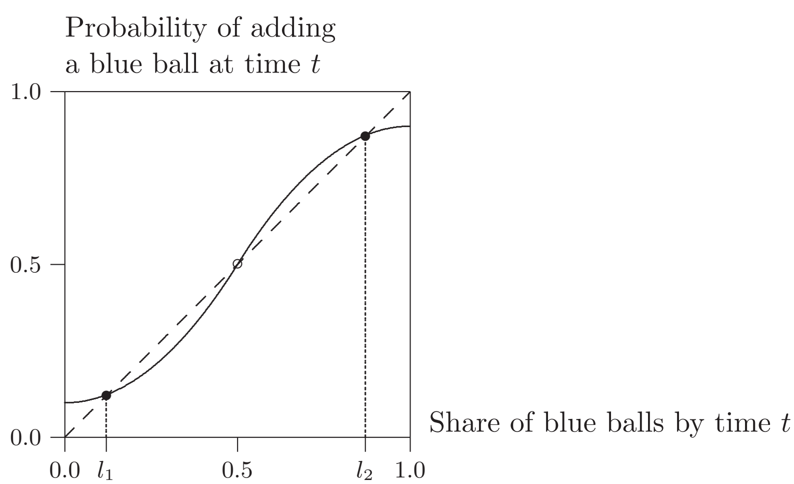

Exhibit 3

Video-cassette recorder market: output levels and VHS's market share. Source: see Endnote 4.
The vertical dashed lines mark the formation of the US Video Software Dealers Association (VSDA).